MATH 245 F22, Exam 2 Solutions

1. Carefully define the following terms: Proof by (vanilla) Induction, Big O.

   To prove $\forall n \in \mathbb{N}, P(n)$ by (vanilla) induction, we must (i) prove $P(1)$; and (2) prove $\forall n \in \mathbb{N}, P(n) \to P(n+1)$. Let a_n, b_n be sequences. We say that a_n is big O of b_n if $\exists n_0 \in \mathbb{N},\ \exists M \in \mathbb{R},\ \forall n \geq n_0,\ |a_n| \leq M|b_n|$.

2. Carefully state the following theorems: Proof by Cases theorem, Division Algorithm theorem.

   To prove $p \to q$ by cases, we must find propositions $c_1, \ldots, c_k$ such that $c_1 \vee \cdots \vee c_k \equiv T$, and prove each of $(p \wedge c_1) \to q, \ldots, (p \wedge c_k) \to q$. The Division Algorithm theorem states: for all integers a, b with $b \geq 1$, there are unique integers q, r such that $a = bq + r$ and $0 \leq r < b$.

3. Prove or disprove: For all $n \in \mathbb{Z}$, we must have $\frac{(n-1)n(n+1)}{3} \in \mathbb{Z}$.

   The statement is true. Let $n \in \mathbb{Z}$ be arbitrary. Applying the Division Algorithm theorem to $n, 3$, we get integers q, r with $n = 3q + r$ and $0 \leq r < 3$. We now have three cases:
   Case $r = 0$: $\frac{(n-1)n(n+1)}{3} = \frac{(n-1)(3q)(n+1)}{3} = (n-1)(q)(n+1) \in \mathbb{Z}$.
   Case $r = 1$: $\frac{(n-1)n(n+1)}{3} = \frac{(3q+1-1)(n)(n+1)}{3} = (q)(n)(n+1) \in \mathbb{Z}$.
   Case $r = 2$: $\frac{(n-1)n(n+1)}{3} = \frac{(n-1)(n)(3q+2+1)}{3} = (n-1)(n)(q+1) \in \mathbb{Z}$.
   In all three cases, we have $\frac{(n-1)n(n+1)}{3} \in \mathbb{Z}$.

   NOTE: We cannot prove this by induction, because the domain is $\mathbb{Z}$.

4. Prove or disprove: For all $x \in \mathbb{R}$, we must have $x\lfloor x \rfloor \leq x\lceil x \rceil$.

   The statement is false, so we need a counterexample. Any negative number that is not an integer will work, but you need to pick a specific one. For example, take $x = -0.5$: $x\lfloor x \rfloor = (-0.5)(-1) = 0.5$, while $x\lceil x \rceil = (-0.5)(0) = 0$. Note that $0.5 > 0$.

5. Prove or disprove: For all $n \in \mathbb{N}$, we must have $5^n > n^2$.

   The statement is true, and the proof will need (vanilla) induction.
   Base case $n = 1$: $5^1 = 5 > 1 = 1^2$. Verified.
   Inductive case: Let $n \in \mathbb{N}$ be arbitrary, and suppose that $5^n > n^2$. Multiply both sides by 5, and we get $5^{n+1} = 5 \cdot 5^n > 5n^2 = n^2 + 2n^2 + 2n^2$. Obviously $n^2 \geq n^2$. Also, since $n \geq 1$, we have $2n^2 \geq 2n$. Lastly, since $n \geq 1$, we have $2n^2 \geq 1$. Adding these, we get $n^2 + 2n^2 + 2n^2 \geq n^2 + 2n + 1 = (n+1)^2$. Combining with the previous, we get $5^{n+1} > (n+1)^2$.

6. Solve the recurrence with initial conditions $a_0 = 3, a_1 = -1$ and relation $a_n = a_{n-1} + 6a_{n-2}$ $(n \geq 2)$.

   The characteristic polynomial is $r^2 - r - 6 = (r-3)(r+2)$. Hence the general solution is $a_n = A3^n + B(-2)^n$. We now apply our initial conditions to get $3 = a_0 = A3^0 + B(-2)^0 = A + B$, and $-1 = a_1 = A3^1 + B(-2)^1 = 3A - 2B$. We solve the linear system $\{A + B = 3, 3A - 2B = -1\}$, getting $A = 1, B = 2$. Hence, our desired specific solution is $a_n = 3^n + 2(-2)^n$.

7. Suppose that an algorithm has runtime specified by recurrence relation $T_n = 9T_{n/3} + n^2$. Determine what, if anything, the Master Theorem tells us.

   This relation is of the type handled by the Master Theorem, with $a = 9, b = 3, k = 2$. We now calculate $d = \log_b a = \log_3 9 = 2$. Because $d = k$, the "middle c_n" case applies, and the theorem tells us that $T_n = \Theta(n^2 \log n)$.

8. Let a_n be a sequence of positive real numbers with $\lim_{n\to\infty} a_n = \infty$. Set $b_n = 1 + a_n$. Prove that $a_n = \Theta(b_n)$.

There are two things to prove:
Proving $a_n = O(b_n)$ (the easier part): Take $n_0 = 1, M = 1$, and let $n \geq n_0$ be arbitrary. We have $|a_n| = a_n \leq 1 + a_n = b_n = M|b_n|$, so $|a_n| \leq M|b_n|$.
Proving $a_n = \Omega(b_n)$ (the harder part): Because $\lim_{n\to\infty} a_n = \infty$, there is some N such that $a_n \geq 1$ for every $n \geq N$. Take $n_0 = N, M = 2$, and let $n \geq n_0$ be arbitrary. We have $M|a_n| = 2a_n = a_n + a_n \geq a_n + 1 = b_n = |b_n|$, so $M|a_n| \geq |b_n|$.

NOTE: The hypothesis $\lim_{n\to\infty} a_n = \infty$ is needed only for part of the proof of $a_n = \Omega(b_n)$ (this part is worth a total of one point). If the sequence a_n did not approach ∞, the statement might be false: e.g. if $a_n = \frac{1}{n}$, then $a_n \neq \Omega(1 + a_n)$.

9. Prove: $\forall x \in \mathbb{R},\ !n \in \mathbb{Z},\ 2n \leq x < 2n + 2$.

Let $x \in \mathbb{R}$ be arbitrary. Suppose $n_1, n_2 \in \mathbb{Z}$ with $2n_1 \leq x < 2n_1 + 2$ and $2n_2 \leq x < 2n_2 + 2$. We recombine these inequalities to get $2n_1 \leq x < 2n_2 + 2$ and $2n_2 \leq x < 2n_1 + 2$. The first one we divide by 2 to get $n_1 < n_2 + 1$. The second we divide by 2 and subtract 1 to get $n_2 - 1 < n_1$. Combining, we get $n_2 - 1 < n_1 < n_2 + 1$. By a theorem from the book (Thm 1.12.d) we conclude $n_1 = n_2$.

10. Prove: $\forall x \in \mathbb{R},\ \exists n \in \mathbb{Z},\ 2n \leq x < 2n + 2$.

Let $x \in \mathbb{R}$ be arbitrary. All we need is to find some $n \in \mathbb{Z}$ making the double inequality true.

METHOD 1: Use Maximal Element Induction. Define $S = \{m \in \mathbb{Z} : m \leq \frac{x}{2}\}$. This set is nonempty, being a half-line, and has upper bound $\frac{x}{2}$. By Maximal Element Induction, S has some maximal element $n \in \mathbb{Z}$, where $n \leq \frac{x}{2}$ and $n+1 > \frac{x}{2}$. Multiply each by 2 and recombine to get $2n \leq x < 2n + 2$, as desired.

METHOD 2: Use Minimal Element Induction. Define $S = \{m \in \mathbb{Z} : m > \frac{x}{2} - 1\}$. This set is nonempty, being a half-line, and has lower bound $\frac{x}{2} - 1$. By Minimal Element Induction, S has some minimal element $n \in \mathbb{Z}$, where $n > \frac{x}{2} - 1$ and $n - 1 \leq \frac{x}{2} - 1$. Multiply each by 2 and recombine to get $2n \leq x < 2n + 2$, as desired.

METHOD 3: Use properties of floors. Take $n = \lfloor \frac{x}{2} \rfloor$, an integer. By the definition of floor, we have $n \leq \frac{x}{2} < n + 1$. Multiply through by 2 to get $2n \leq x < 2n + 2$, as desired.

METHOD 4 (found by a clever student): Take $t = \lfloor x \rfloor$, an integer. By definition of floor, we have $t \leq x < t + 1$. By a theorem from the book (Thm 1.6), t is either even or odd.
Case t is even: There is $n \in \mathbb{Z}$ with $t = 2n$. Hence $2n = t \leq x < t + 1 = 2n + 1 < 2n + 2$.
Case t is odd: There is $n \in \mathbb{Z}$ with $t = 2n + 1$. Hence $2n < 2n + 1 = t \leq x < t + 1 = (2n + 1) + 1 = 2n + 2$.
In both cases, we have found some integer n with $2n \leq x < 2n + 2$.